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Exponential Decay of Connectivities in the Two-Dimensional Ising Model

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We prove some results concerning the decay of connectivities in the low-temperature phase of the two-dimensional Ising model. These provide the bounds necessary to establish, nonperturbatively, large-deviation properties for block magnetizations in these systems. We also obtain estimates on the rate at which the finite-volume, plus-boundary-condition expectation of the spin at the origin converges to the spontaneous magnetization.

KEY WORDS: Two-dimensional Ising model; percolation; exponential decay; FK representation; correlation lengths; large deviations.

1. INTRODUCTION

We consider the two-dimensional nearest neighbor ferromagnetic Ising model. Our main result is that, in the + state, the connectivity function for $(-*)$ percolation tends to zero exponentially rapidly at all temperatures below T_c. This strengthens the celebrated result of Russo,[9] who proved that below T_c, in the + state, the plus spins percolate and there is almost surely no $(-*)$ infinite cluster. Related, but somewhat weaker results about connectivities have recently been established in Ref. 4.

Our principal motivation comes from Ref. 10, where large-deviation properties of the average magnetizations of large blocks in the + phase were investigated. It was demonstated in Ref. 10 that exponential decay of the $(-*)$ connectivity is sufficient to establish that these large deviations of the block magnetizations behave as the surface area of the blocks. (The estimates in Ref. 4 and those implicit in Ref. 9 are insufficient for these

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purposes.) For low enough temperatures, Peierls types of estimates can be used to show this exponential decay. Our result extends the above results of Ref. 10 to all temperatures below T_c. In addition, Ref. 10 derived large-deviation estimates for finite systems with free boundary conditions under the assumption of low enough temperatures; here, we extend these estimates to the full phase coexistence regime.

Finally, as a bonus, we obtain essentially optimal estimates on the rate of convergence of the finite-volume, +-boundary-condition, single-site magnetization to the infinite-volume (spontaneous) magnetization.

Before making any precise statements, let us introduce some notation. If $\Gamma \subset \mathbb{Z}^2$, then $|\Gamma|$ will denote its cardinality (number of sites), and $\partial\Gamma$ its (interior) boundary, i.e., the set of sites in Γ that have a neighbor in $\mathbb{Z}^2 \backslash \Gamma$. A chain in $\mathbb{Z}^2$ is a sequence $\{x_1,..., x_n\}$, with no repeats, such that x_i and x_{i+1} are nearest neighbors for $i = 1,..., n-1$. A circuit is a chain such that x_1 and x_n are also nearest neighbors. Two distinct sites are said to be $(*)$ adjacent if they are nearest or next nearest neighbors. In the obvious way, one may then define $(*)$ chains and $(*)$ circuits. Given an Ising spin configuration $\sigma \in \{-1, +1\}^{|\Gamma|}$, a $(+)$ chain is a chain in Γ such that $\sigma_x = +1$ for each x belonging to the chain. The $(+)$ circuits, $(+*)$ chains, etc., are defined analogously. Two sites, x and y, are said to be $(+)$ connected if they belong to the same $(+)$ chain. Similar definitions describe the notions of $(-)$ connections, $(-*)$ connections, etc.

Given a $\Gamma \subset \mathbb{Z}^2$, $|\Gamma| < \infty$, for fixed inverse temperature β and boundary condition $a \equiv \{a_x \mid x \in \mathbb{Z}^2 \backslash \Gamma\}$, the Gibbs state in Γ is given by

$$\mu_{\Gamma,a,\beta}(\sigma) = \frac{1}{\mathscr{Z}} \exp\left[\frac{1}{2}\beta\left(\sum_{\substack{\langle x,y\rangle \\ x,y \in \Gamma}} \sigma_x \sigma_y + \sum_{\substack{\langle x,y\rangle \\ x \in \Gamma \\ y \notin \Gamma}} \sigma_x a_y\right)\right] \tag{1}$$

where $\langle x, y\rangle$ means that x and y are nearest neighbors and each pair should be considered only once in the sum. The normalization constant $\mathscr{Z}$ is called the partition function. The + boundary conditions are defined by fixing $a_x = 1$ for all $x \in \mathbb{Z}^2 \backslash \Gamma$, while free boundary conditions are obtained by fixing the $a_x = 0$. These two boundary conditions will be indicated by + or 0, respectively, in place of the subscript a above. We will also use the notation $\langle \text{-} \rangle_{\Gamma,a,\beta}$ to indicate the expectation of functions with respect to the $\mu_{\Gamma,a,\beta}$ measure.

It is well known that as Γ increases to $\mathbb{Z}^2$, $\mu_{\Gamma,+,\beta}$ converges (in the distributional sense) to the infinite-volume measure $\mu_{+,\beta}$ called the +phase; $\mu_{-,\beta}$ is the analogous −phase. Let $m(\beta)$ be the expected value of σ_0 in the +phase; $m(\beta)$ is called the spontaneous magnetization. We let $\beta_c = \inf\{\beta \geqslant 0 \mid m(\beta) > 0\}$ denote the critical temperature.

The $(-*)$ connectivity function for the $+$phase is defined by

$$f_I(\beta, \eta) = \mu_{+,\beta}(T_{0n}^{-*}) \tag{2}$$

where T_{0n}^{-*} is the event that $(0, 0)$ is $(-*)$ connected to $(0, n)$. The usual supermultiplicity arguments—which here involve the FKG inequalities for $\mu_{+,\beta}(\cdot)$—imply the existence of

$$\lim_{n \to \infty} [-n^{-1} \log f_I(\beta, n)] \equiv \gamma_I(\beta) \in [0, \infty) \tag{3}$$

In Ref. 10 the following facts were established: Let $\Gamma \subset \mathbb{Z}^2$ with $|\Gamma| < \infty$, set

$$M_\Gamma(\sigma) = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} \sigma_x \tag{4}$$

and consider the sequence of squares (the choice here is somewhat different from that used in Ref. 10): $\Lambda_n = \{x \in \mathbb{Z}^2 \mid -n \leqslant x_1, x_2 \leqslant +n\}$. Then:

S1. Assume $\beta > \beta_c$. Provided that the $(-*)$ connectivity function for $\mu_{+,\beta}$ decays exponentially (i.e., $\gamma_I > 0$), then if $-1 \leqslant a < b$ with $|b| < m(\beta)$, there are nontrivial constants A_1, A_2, C_1, C_2 such that

$$A_1 e^{-c_1 n} \leqslant \mu_{+,\beta}\{M_{\Lambda_n} \in (a, b)\} \leqslant A_2 e^{-c_2 n} \tag{5}$$

S2. Provided that $\beta > \log 3$, then if $a < b$ with $|a|$, $|b| < m(\beta)$, there are nontrivial constants A_1, A_2, C_1, C_2 such that

$$A_1 e^{-c_1 n} \leqslant \mu_{\Lambda_n,0,\beta}\{M_{\Lambda_n} \in (a, b)\} \leqslant A_2 e^{-c_2 n} \tag{6}$$

Remark. Note that although the large-deviation estimates (5) and (6) are optimal in the sense that the exponential decay rate n is the surface area of the cube Λ_n, they do not follow from optimal hypotheses. Indeed, the condition for S1 implies $\beta > \beta_c$ (by Ref. 9), but is not necessarily implied by it; the condition for S2, $e^{-\beta} < 1/3$, is the well-known Peierls estimate on β_c. According to physical intuition, the input hypotheses for S1 and S2 ought, simply, to be the condition $\beta > \beta_c$; this is what we will show below.

In particular, we will show that exponential decay of $(-*)$ connectivity in the $+$phase is equivalent to positive spontaneous magnetization:

Theorem 1. For any $\beta > \beta_c$, $f_I(\beta, n)$ decays exponentially in n, i.e., $\gamma_I > 0$.

Although Theorem 1 is of some intrinsic interest, its principal benefit is:

Corollary 1. The behavior described in S1 holds for all $\beta > \beta_c$.

The problem of removing the explicit Peierls estimate on β_c for S2 is handled by techniques related to the proof of Theorem 1. The upshot is:

Theorem 2. The behavior described in S2 holds for all $\beta > \beta_c$.

With only a little more work, we are able to obtain fairly accurate bounds on the rate of convergence of single-site magnetizations in finite-volume, +boundary condition systems to $m(\beta)$:

Theorem 3. For $\beta > \beta_c$, there are nontrivial constants A_a, A_b and γ_a, γ_b such that

$$A_a \exp(-\gamma_a n) \leqslant \langle \sigma_0 \rangle_{A_n, +, \beta} - m(\beta) \leqslant A_b \exp(-\gamma_b n)$$

A central ingredient in our analysis will be the Fortuin and Kasteleyn (FK) random cluster representation, introduced in Ref. 3, which has been studied in Ref. 1. In the next section we review the relevant facts about this representation of the Ising spin system. In the final section we prove the above theorems.

2. PRELIMINARIES

Given $\Gamma \subset \mathbb{Z}^2$, let $\mathscr{L}_\Gamma$ denote the set of unoriented pairs of nearest neighbor sites of $\mathbb{Z}^2$ such that at least one member of the pair is in Γ. The elements of $\mathscr{L}_\Gamma$ are called bonds. Two distinct bonds are said to be adjacent if they share a site. A chain of bonds in $\mathscr{L}_\Gamma$ is a sequence $(b_1,..., b_n \mid b_i \in \mathscr{L}_\Gamma)$ such that b_i and b_{i+1} are adjacent for $i = 1,..., n-1$. If b_n and b_1 are also adjacent, we have a circuit of bonds. (One may add the word "self-avoiding" to the above definitions if no sites in the above chains occur more than twice.) Two sites $x, y \in \Gamma$ are said to be connected by a chain of bonds if they both belong to a bond of the chain.

The random cluster probability measure is described by assigning to each element in $\mathscr{L}_\Gamma$ a variable 0 (the bond is vacant) or 1 (the bond is occupied) in such a way that the probability of a configuration $\omega \in \{0, 1\}^{|\mathscr{L}_\Gamma|}$ is proportional to

$$p^{\mathscr{b}(\omega)}(1-p)^{v(\omega)} q^{c(\omega)} \tag{7}$$

Here, p is a number in $(0, 1)$, while $q \in (0, \infty)$; $\mathscr{b}(\omega)$ is the number of occupied bonds in $\mathscr{L}_\Gamma$, $v(\omega)$ the number of vacant bonds, and $c(\omega)$ the number of distinct clusters in Γ induced by ω. In fact, we will have to consider two different ways of counting $c(\omega)$, according to different boundary conditions:

(i) *Free boundary conditions*: Two sites are said to belong to the same cluster if and only if they are connected by an occupied chain of bonds. In this case we will write $c_f(\omega)$ for the number of clusters, and denote by $P_{\Gamma,f,p,q}(\cdot)$ the random cluster probability measure.

(ii) *Wired boundary conditions*: Two sites are said to belong to the same cluster either if they are connected according to the above definition or if they are both connected to a site in $\mathbb{Z}^2\backslash\Gamma$ by chains of occupied bonds. In this case we will write $c_w(\omega)$ for the number of clusters and denote by $P_{\Gamma,w,p,q}(\cdot)$ the random cluster probability measure.

Remark. The integer values for q in these random cluster measures provide representations of the q-state Potts models. Here, our interest focuses on the case $q=2$, which corresponds to the Ising model. We will henceforth take $q=2$, and omit q from all subsequent expressions.

The basic relations that we will need between the Gibbs and the random cluster measures are the following: Set $p=1-e^{-\beta}$. Then:

P1. $\langle\sigma_x\rangle_{\Gamma,+,\beta}=P_{\Gamma,w,p}\{x$ belongs to the cluster of the boundary of $\Gamma\}$.

P2. $\langle\sigma_x\sigma_y\rangle_{\Gamma,a,\beta}=P_{\Gamma,b,p}\{x$ & y belong to the same cluster$\}$, where a is $+$ and b is w, or a is 0 and b is f.

P3. For $\mathscr{R}\subset\Gamma$, a and b as in P2

$$\mu_{\Gamma,a,\beta}\{\text{all the spins in } \mathscr{R} \text{ are identical}\}$$
$$\geqslant P_{\Gamma,b,p}\{\text{all sites in } \mathscr{R} \text{ belong to the same cluster}\}$$

P4. For $\mathscr{R}\subset\Gamma$,

$$\mu_{\Gamma,+,\beta}\{\text{all the spins in } \mathscr{R} \text{ are plus}\}$$
$$\geqslant P_{\Gamma,w,p}\{\text{all sites in } \mathscr{R} \text{ belong to the cluster of the boundary of } \Gamma\}$$

P5. The measures $P_{\Gamma,w,p}(\cdot)$ and $P_{\Gamma,f,p}(\cdot)$ satisfy the FKG inequalities with respect to the usual partial order on the bond configurations.

P6. Unique infinite-volume limits exist for the $P_{\Gamma,w,p}(\cdot)$ and $P_{\Gamma,f,p}(\cdot)$. Furthermore, all of the above statements, modified where necessary, hold in the infinite-volume limit.

Explicit proofs of P1–P6, and/or sufficient background material for the reader to provide proofs, can be found in Ref. 1.

Next, we discuss some basic facts about duality for the two-dimensional random cluster models. (For more on this, see Ref. 12, Section II.)

As usual, the dual lattice consists of the pairs of nearest neighbors in $\mathbb{Z}^{*2} \equiv (\mathbb{Z}+1/2)^2$. For each bond $\{x, y\}$ in $\mathscr{L}_\Gamma$, there is a unique corresponding dual bond $\{z, t\}$ defined by the property that the two bonds intersect at their midpoints. Let $\mathscr{L}^*_\Gamma$ denote the set of bonds dual to the bonds of $\mathscr{L}_\Gamma$. Given an $\omega \in \{0, 1\}^{|\mathscr{L}_\Gamma|}$, we may define an $\omega^* \in \{0, 1\}^{|\mathscr{L}_\Gamma^*|}$ by saying that a dual bond is vacant if and only if the corresponding direct bond is occupied. Obviously, in this way, the random cluster measures on $\{0, 1\}^{|\mathscr{L}_\Gamma|}$ induce (dual) measures on $\{0, 1\}^{|\mathscr{L}_\Gamma^*|}$. What is not so obvious is that these dual measures are themselves random cluster measures. Let us pause to sketch a proof of this fact.

Below, K will represent a "constant," but its value will change from expression to expression. (Typically, it will be equal to $|\Gamma|$, $|\mathscr{L}_\Gamma|$, etc.) We will use N to represent a normalization "constant," the value of which will also change.

Let $l_f(\omega)$ and $l_w(\omega)$ be the number of independent loops of occupied bonds in a configuration $\omega \in \{0, 1\}^{|\mathscr{L}_\Gamma|}$ with free or wired boundary conditions, respectively. The following facts can be proved by induction (or taken as a definition):

$$c_f(\omega) = K - \ell(\omega) + l_f(\omega) \tag{8a}$$

$$c_w(\omega) = K - \ell(\omega) + l_w(\omega) \tag{8b}$$

Since $\ell(\omega) + v(\omega)$ is a constant, the factor of $(1-p)^{v(\omega)}$ in (7) can be replaced by $(1-p)^{-\ell(\omega)}$ by readjustment of the normalizing constant. Therefore

$$\begin{aligned} P_{\Gamma,f,p}(\omega) &= N\left[\frac{p}{1-p}\right]^{\ell(\omega)} 2^{K-\ell(\omega)+l_f(\omega)} \\ &= N\left[\frac{p}{2(1-p)}\right]^{\ell(\omega)} 2^{l_f(\omega)} \\ &= N\left[\frac{\tilde{p}}{1-\tilde{p}}\right]^{\ell(\omega)} 2^{l_f(\omega)} \end{aligned} \tag{9}$$

where $\tilde{p} = p/(2-p)$. Analogously,

$$P_{\Gamma,w,p}(\omega) = N\left[\frac{\tilde{p}}{1-\tilde{p}}\right]^{\ell(\omega)} 2^{l_w(\omega)} \tag{10}$$

Let $l_f^*(\omega)$ and $l_w^*(\omega)$ be the number of independent loops of occupied dual bonds in the configuration ω^* with free or wired boundary conditions, respectively. (To simplify what is to follow, let us assume that the set Γ is chainwise connected.) Then the following relations are well known or can easily be proved:

$$c_w(\omega) = l_f^*(\omega) + 1 \tag{11a}$$

$$c_f(\omega) = l_w^*(\omega) + 1 \tag{11b}$$

Furthermore, if $\ell^*(\omega)$ is the number of occupied (dual) bonds of ω^*, then clearly

$$\ell^*(\omega) = K - \ell(\omega) \tag{12}$$

Therefore

$$\begin{aligned} P_{\Gamma,w,p}(\omega) &= N\left[\frac{p}{1-p}\right]^{\ell(\omega)} 2^{c_w(\omega)} \\ &= N\left[\frac{1-p}{p}\right]^{\ell^*(\omega)} 2^{l_f^*(\omega)} \end{aligned} \tag{13a}$$

and

$$P_{\Gamma,f,p}(\omega) = N\left[\frac{1-p}{p}\right]^{\ell^*(\omega)} 2^{l_w^*(\omega)} \tag{13b}$$

Comparison of (9) and (10) with (13a) and (13b) shows that if ω is distributed according to $P_{\Gamma,w,p}$ (resp. $P_{\Gamma,f,p}$), then ω^* is distributed according to P_{Γ,f,p^*} (resp. P_{Γ,w,p^*}), where $p^* \equiv 1 - \tilde{p}$. Observe that the boundary conditions are interchanged and the corresponding inverse temperature β^* satisfies the Kramers–Wannier relation

$$e^{-\beta^*} = \mathrm{th}(1/2\beta) \tag{14}$$

An important fact that we will need later is that if $\beta > \beta_c$, then $\beta^* < \beta_c$. (The self-dual point is, of course, β_c.) Now consider the correlation function $\langle \sigma_0 \sigma_n \rangle_{\beta^*}$, where the expected value corresponds to the (unique) infinite-volume state at inverse temperature β^*. From the usual subadditivity arguments (which here invoke the GKS inequality), it follows that the decay rate

$$\gamma_\Pi(\beta^*) = -\lim_{n \to \infty} \frac{1}{n} \log \langle \sigma_0 \sigma_n \rangle_{\beta^*} \tag{15}$$

is well-defined. It is known from Onsager's exact solution[8] (as remarked in Section II of Ref. 5) that for any $\beta^* < \beta_c$, $\langle \sigma_0 \sigma_n \rangle_{\beta^*}$ decays exponentially in n, or, in other words, γ_Π is positive. By P2 and the duality relations described above, this also tells us that when $\beta > \beta_c$, the dual bonds in the random cluster problem have connectivities that decay exponentially fast. This is the central ingredient in all of our proofs.

3. PROOFS OF THE THEOREMS

Our strategy, roughly speaking, will be to transform our problems in the spin systems to problems that may be addressed in the FK representation. Then, using the connection between the corresponding *dual* models, the desired estimates are obtained from knowledge of the *high*-temperature properties of the Ising model.

Before proceeding further, we must attend to the detail of extending (15) to points that are not along a lattice axis.

Lemma 1. For all $\beta < \beta_c$,

$$\langle \sigma_x \sigma_y \rangle_\beta \leqslant \exp[-\gamma_\Pi(\beta)\, d(x, y)]$$

where $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Proof. A number of results of this sort have appeared in the literature at various times. (See, e.g., Ref. 7, Proposition 1.) The only necessary observation is that if $(z^{(n)}) = (z_1{}^{(n)}, z_2{}^{(n)})$ is a sequence of lattice points with $z_2{}^{(n)}/z_1{}^{(n)} \equiv \tan\theta$ fixed, then

$$\gamma_\Pi(\beta, \theta) = -\lim_{n\to\infty} \frac{|\log\langle \sigma_0 \sigma_{z^{(n)}} \rangle_\beta|}{|z_1{}^{(n)}|} \tag{16}$$

exists and satisfies

(i) $\langle \sigma_0 \sigma_{z^{(n)}} \rangle_\beta \leqslant \exp[-z_1^{(n)} \gamma_\Pi(\beta, \theta)]$.

(ii) $\xi_0(\theta)$ is a convex, symmetric function of $\tan\theta$.

We may halt these considerations at $\theta = 45°$. ∎

In what follows, we could work directly with the various limiting infinite-volume measures. However, some of the details are conceptually simpler if we work in finite-volume systems. Thus, we will typically consider large squares Λ_k and obtain estimates uniform in k once k is large enough. To simplify future notation, we will use the index k instead of Λ_k to identify measures, expectations, etc.; $*k$ will denote the sites of $\mathbb{Z}^{2*}$ that are dual to those in Λ_k.

Proof of Theorem 1. We claim that the intersection of two events in the wired random cluster system (both of which are very likely) is representative of a subclass of the spin configurations in which T_{0n}^{-*} does not occur. The first event, which will be called $\mathscr{R}_n$, is the occurrence, in the annulus $\Omega_n = \Lambda_n \backslash \Lambda_{[n/2]}$, of a circuit of occupied bonds. The second event, denoted by $\mathscr{M}_n$, is the event that at least one site in $\Lambda_{[n/2]}$ belongs to the boundary cluster of the system.

Now, a circuit of + spins in Ω_n will prevent T_{0n}^{-*} from occurring. However, although the event $\mathscr{R}_n$ produces a circuit of spins of the same type, it is not necessarily the case that these will be of the plus type. The simultaneous occurrence of $\mathscr{M}_n$ provides this guarantee. Thus we have, using P3 and P4,

$$\begin{aligned} \mu_{k,+,\beta}(T_{0n}^{-*}) &\leqslant 1 - P_{k,w,p}(\mathscr{M}_n \cap \mathscr{R}_n) \\ &\leqslant P_{k,w,p}(\mathscr{M}_n^c) + P_{k,w,p}(\mathscr{R}_n^c) \end{aligned} \tag{17}$$

Let us exploit the duality to show that the rhs of (17) is very small. In order for $\mathscr{R}_n$ to fail, there must be an occupied chain of dual bonds that connects the two (disconnected) pieces $(\partial\Omega^*)_1$ and $(\partial\Omega^*)_2$ of $\partial\Omega^*$. Then, by the duality relations and P2,

$$\begin{aligned} P_{k,w,p}(\mathscr{R}_n^c) &\leqslant \sum_{\substack{x \in (\partial\Omega)_1 \\ y \in (\partial\Omega)_2}} P_{*k,f,p^*}\,(x \text{ and } y \text{ belong to the same dual cluster}) \\ &\leqslant \sum_{\substack{x \in (\partial\Omega)_1 \\ y \in (\partial\Omega)_2}} \langle \sigma_x \sigma_y \rangle_{*k,0,\beta^*} \end{aligned} \tag{18}$$

Similarly, in order for $\mathscr{M}_n$ not to occur, there must be an occupied dual circuit that separates $\Lambda_{[n/2]}$ from $\partial\Lambda_n$. This would imply the existence of an occupied dual chain connecting the sets

$$\begin{aligned} S_1 &= \{x \in (\mathbb{Z} + 1/2)^2 \mid x_1 = 1/2,\ x_2 > 1/2n\} \\ S_2 &= \{x \in (\mathbb{Z} + 1/2)^2 \mid x_1 = 1/2,\ x_2 < -1/2n\} \end{aligned} \tag{19}$$

As in the derivation of Eq. (18), we obtain

$$P_{k,w,p}(\mathscr{M}_n^c) \leqslant \sum_{\substack{x \in S_1 \\ y \in S_2}} \langle \sigma_x \sigma_y \rangle_{*k,0,\beta^*} \tag{20}$$

From the GKS inequality (or the FKG inequality in the FK representation), the finite-volume free boundary correlation functions are bounded above by their infinite-volume analogs. Thus

$$\langle \sigma_x \sigma_y \rangle_{*k,0,\beta^*} \leqslant \langle \sigma_x \sigma_y \rangle_{\beta^*} \leqslant \exp[-\gamma_\Pi\, d(x, y)] \tag{21}$$

From this it follows that $\mu_{k,+,\beta}(T_{0n}^{-*})$ decays with n as desired. Furthermore, we have shown that $\gamma_{\mathrm{I}}(\beta) \geqslant 1/2\gamma_{\mathrm{II}}(\beta^*)$. ∎

Proof of Theorem 2. We adapt the proof of Theorem 4 in Ref. 10. The basic argument is that, close to the boundary of Λ_n, with overwhelming probability, there is either a circuit of + spins or a circuit of − spins. Therefore, inside this circuit, the state is roughly $\mu_{+,\beta}$ or $\mu_{-,\beta}$—indeed, the restrictions of the free boundary measures to events that take place inside the rings FKG dominate the corresponding infinite-volume Gibbs measures. From these facts and the large-deviation results for $\mu_{+,\beta}$ and $\mu_{-,\beta}$ (Theorem 1 in Ref. 10; Corollary 1 in the present paper), the proof follows easily.

For $0 < \delta < 1$ fixed, let Ω_n^δ denote the annulus $\Lambda_n \backslash \Lambda_{n(1-\delta)}$, and define the event

$$\mathscr{W}_n^\delta = \{\omega \mid \exists \text{a} + \text{or a} - \text{circuit in } \Omega_n^\delta \text{ that surrounds } \Lambda_{n(1-\delta)}\} \tag{22}$$

From the proof of Theorem 4 in Ref. 10 it is sufficient to show that there are nontrivial constants A and c such that

$$\mu_{n,0,\beta}((\mathscr{W}_n^\delta)^c) \leqslant A e^{-cn} \tag{23}$$

Let us shift attention to the FK representation. Consider the event $\mathscr{N}_n^\delta$ that in the annulus Ω_n^δ there is an occupied circuit of bonds surrounding $\Lambda_{n(1-\delta)}$. By P3,

$$\mu_{n,0,\beta}((\mathscr{W}_n^\delta)^c) \leqslant P_{n,f,p}((\mathscr{N}_n^\delta)^c) \tag{24}$$

However, if $\mathscr{N}_n^\delta$ does not occur, there must be an occupied chain of dual bonds connecting the two disconnected pieces ($R_1 = \partial\Lambda_n^*$ and $R_2 = \partial\Omega_n^{\delta*} \backslash \partial\Lambda_n^*$) of the boundary of $\partial\Omega_n^{\delta*}$. By duality and P1,

$$\begin{aligned} P_{n,f,p}((\mathscr{N}_n^\delta)^c) &\leqslant \sum_{x \in R_2} P_{*n,w,p^*}\{x \text{ belongs to the cluster of the boundary of } \Lambda_n^*\} \\ &\leqslant \sum_{x \in R_2} \langle \sigma_x \rangle_{*n,+,\beta^*} \end{aligned} \tag{25}$$

Following Ref. 2, we will use the Simon–Lieb inequality[11,6] to finish the proof. The (fixed) plus spins that form the boundary condition of $\mu_{*n,+,\beta^*}$ can be regarded as a single (giant) fixed spin $\sigma_{\mathbb{B}}$ also set to plus—albeit in a somewhat contorted lattice. Let us now consider the system where $\sigma_{\mathbb{B}}$ is allowed dynamic degrees of freedom, but otherwise has *no* boundary condition. We will denote thermal expectations in this measure by $\langle - \rangle_{*n, \mathbb{B}, \beta^*}$. It is not terribly difficult to see that

$$\langle \sigma_x \rangle_{*n, +, \beta^*} = \langle \sigma_x \sigma_{\mathbb{B}} \rangle_{*n, \mathbb{B}, \beta^*} \tag{26}$$

Using (26) and the Simon–Lieb inequality, we have

$$\begin{aligned} \langle \sigma_x \rangle_{*n, +, \beta^*} &\leqslant \sum_{y \in \partial \Lambda_n^*} \langle \sigma_x \sigma_y \rangle_{*n, 0, \beta^*} \langle \sigma_y \sigma_{\mathbb{B}} \rangle_{*n, \mathbb{B}, \beta^*} \\ &\leqslant \sum_{y \in \partial \Lambda_n^*} \langle \sigma_x \sigma_y \rangle_{*n, 0, \beta^*} \\ &\leqslant \sum_{y \in \partial \Lambda_n^*} \langle \sigma_x \sigma_y \rangle_{\beta^*} \end{aligned} \tag{27}$$

where the last step is from the aforementioned dominance. Combining Eq. (27) with the exponential decay implied by Lemma 1 and the fact that $\beta^* < \beta_c$, one obtains the desired result. ∎

As a consequence of the above theorems—and the techniques used to prove them—we have the following additional, intuitive result:

Proof of Theorem 3. Let $\beta > \beta_c$ and consider the event $\mathscr{R}_n^{-*}$ that there is a $(^{-*})$ ring surrounding the box Λ_n. We have

$$\begin{aligned} \langle \sigma_0 \rangle_{n, +, \beta} - m(\beta) = {} & \mu_{+, \beta}(\mathscr{R}_n^{-*}) [\langle \sigma_0 \rangle_{n, +, \beta} - \langle \sigma_0 \mid \mathscr{R}_n^{-*} \rangle_{+, \beta}] \\ & + [1 - \mu_{+, \beta}(\mathscr{R}_n^{-*})] [\langle \sigma_0 \rangle_{n, +, \beta} - \langle \sigma_0 \mid (\mathscr{R}_n^{-*})^c \rangle_{+, \beta}] \end{aligned} \tag{28}$$

Notice that, by the FKG inequality, the second term is positive. Indeed, the absence of a $(^{-*})$ ring surrounding Λ_n is not as positive an incentive as plus boundary conditions on Λ_n. Thus,

$$\langle \sigma_0 \rangle_{n, +, \beta} - m(\beta) \geqslant \mu_{+, \beta}(\mathscr{R}_n^{-*}) [\langle \sigma_0 \rangle_{n, +, \beta} - \langle \sigma_0 \mid \mathscr{R}_n^{-*} \rangle_{+, \beta}] \tag{29}$$

Clearly, all remaining terms are positive. In particular, the condition $\mathscr{R}_n^{-*}$ interdicts the + state inside Λ_n and we have

$$\langle -\sigma_0 \mid \mathscr{R}_n^{-*} \rangle_{+, \beta} \geqslant m(\beta) \tag{30}$$

so that

$$\langle \sigma_0 \rangle_{n, +, \beta} - m(\beta) \geqslant 2m(\beta)\, \mu_{+, \beta}(\mathscr{R}_n^{-*}) \tag{31}$$

We claim that, in fact, $\mu_{+, \beta}(\mathscr{R}_n^{-*}) \geqslant \exp(-8n\gamma_{\mathrm{I}})$, where it is noted that $8n$ is the perimeter of the box. (An upper bound of this form can also be

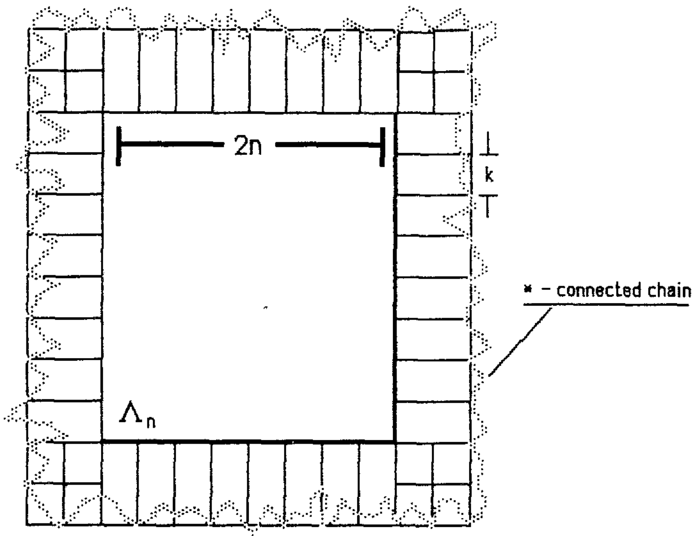

Fig. 1. Construction of the event $\mathscr{R}_n$.

obtained.) To this end, consider the event $T^{-*}_{k,\square} \subset T^{-*}_{0k}$ in which there is a $(^{-*})$ chain from the origin to $(0, k)$ that does not pass to the right of the line $x_1 = 2k$. By the reasoning of Lemma 1, it is seen that $\mu_{+,\beta}(T^{-*}_{k,\square})/\mu_{+,\beta}(T^{-*}_{0k})$ tends to unity exponentially rapidly.

We now surround the region Λ_n with a mesh of strips of scale k. (See Fig. 1.) Estimating from below the probability of on the order of $8n/k$ events that are translates and rotations of $T^{-*}_{k,\square}$, we obtain

$$\mu_{+,\beta}(\mathscr{R}_n^{-*}) \geqslant \exp[-\gamma_{\mathrm{I}}(\beta)\, 8n(1+\varepsilon)] \tag{32}$$

for any ε, provided n and k are large enough. This, together with (31), provides the desired lower bound.

The upper bounds are obtained in a similar fashion. Indeed, consider the event $T^{-*}_{\partial\Lambda_n}$, that the origin is $(^{-*})$ connected to the boundary of Λ_n. A derivation similar to that above shows

$$\langle \sigma_0 \rangle_{n,+,\beta} - m(\beta) \leqslant 2\mu_{+,\beta}(T^{-*}_{\partial\Lambda_n}) \tag{33}$$

Using subadditivity and the exponential decay of T^{-*}_{0n}, we obtain the desired result. ∎

Remark. It is worth noting that all of the γ's in this work may be identified—in the scaling sense—with multiples of the inverse of the correlation length.

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